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APPROXIMATE COMPUTATION OF THE LEAST GUARANTEED ESTIMATE IN LINEAR DIFFERENTIAL GAMES WITH A FIXED DURATION*

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A method for the approximate computation of the least guaranteed game estimate is constructed for linear fixed-duration differential games, and an estimate of its rate of convergence is given. The paper is closely related to /1—9/.

The motion of an n-dimensional vector $z \in R^n$ is described by the equation

$$z^{\cdot} = A(t)z + u + v, \quad z(t_0) = z_0; \; t \in I = [t_0, T] \; (t_0 < T) \tag{1}$$

$$u \in P(t) \subset R^n, \quad v \in Q(t) \subset R^n; \quad |P(t)| \leqslant a_1(t), \quad |Q(t)| \leqslant a_2(t) \tag{2}$$

The elements of the n th-order square $A(t)$ are defined and are Lebesgue-summable on I; $P(t)$ and $Q(t)$ are nonempty compacta for each $t \in I$ and they depend measurably on $t \in I$ (see /10/) and satisfy the stated conditions; moreover $|X| = \max_{x \in X} |x|$ for a nonempty compactum $X \subset R^n$, $a_i(t)$ is a Lebesgue-summable function on I. The performance of the pair of measurable functions $u(t) \in P(t), v(t) \in Q(t), t \in I$, is estimated by the quantity $\varphi(z(T))$ ($\varphi(z)$ is a scalar function continuous on R^n). The first player deals with the choice of u and strives to minimize $\varphi(z(T))$. The second player deals with the choice of vector v and strives to maximize $\varphi(z(T))$. The second player selects the measurable control $v(t) \in Q(t)$ as a program control on I. The measurable control $u(t) \in P(t)$ is at the first player's disposal and is constructed for $t \in I$ on the basis of knowing Eq.(1), the initial state $z(t_0) = z_0$ and the control $v(s)$ for $t_0 \leqslant s \leqslant t$, in the form $u(t) = U(t, v_t(\cdot))$, where $v_t(\cdot)$ denotes the function $v(s), t_0 \leqslant s \leqslant t$, while the mapping U is defined on the set of measurable functions $v(t) \in Q(t), t \in I$, and maps such functions $v(\cdot)$ into the set of measurable functions $u(t) \in P(t)$, $t \in I$.

Game (1) is examined from the first player's viewpoint. It is assumed that he knows Eq. (1), the vector z_0, the function φ and the control $v_t(\cdot)$ for each $t \in I$. It is natural to characterize the performance of the first player's given strategy U by the quantity $\sup_{v(\cdot)} \varphi(z(T))$, where $z(T)$ (see /1/) corresponds to the measurable controls $v(t), u(t) = U(t, v_t(\cdot)), t \in I$. An important characteristic of the first player's capabilities is the quantity

$$\gamma = \inf_{U} \sup_{v(\cdot)} \varphi(z(T)) \tag{3}$$

which is called the least guaranteed estimate. The computation of the quantity γ causes great difficulty. Therefore, an approximate computation of γ to any preassigned accuracy is of interest.

By $\Phi(t, s)$ $(t_0 \leqslant s \leqslant t \leqslant T)$ we denote the matrizant (see /11/) of the homogeneous equation $x^{\cdot} = A(t)x$. We note that for fixed measurable $u(t) \in P(t), v(t) \in Q(t), t \in I$, the Cauchy formula

$$z(t) = \Phi(t, t_0)z_0 + \int_{t_0}^{t} \Phi(t, s)(u(s) + v(s))\, ds \tag{4}$$

is valid for the solution of Eq.(1). We set

$$D = \Phi(T, t_0)z_0 + \int_{t_0}^{T} \Phi(T, s)(P(s) + Q(s))\, ds \tag{5}$$

where the integral is understood in the sense usual for the theory of multivalued mappings (see /10/) and the plus sign signifies the algebraic addition of sets. It can be proved that D is a nonempty convex compactum.

On $(T, 2T - t_0]$ we define the matrix-valued function $A(t)$ (see (1)) as an n th-order null square matrix. Now the matrizant $\Phi(t, s)$ is defined for $t_0 \leqslant s \leqslant t \leqslant 2T - t_0$. The scalar product in R^n of arbitrary vectors a and b is defined by the formula $(a, b) = a_1 b_1 + \ldots + a_n b_n$, where a_i, b_i are the coordinates of vectors a, b. We note the relation ($|\cdot|$ is the operator norm of a matrix)

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$$\begin{aligned}
&|\Phi(t,s)| \leqslant E(s,t), \quad t_0 \leqslant s \leqslant t \leqslant 2T - t_0 \\
&|\Lambda^{-1}(t)| \leqslant E(t_0,t), \quad t_0 \leqslant t \leqslant 2T - t_0 \\
&E(\alpha,\beta) = \exp \int_\alpha^\beta |A(r)|\, dr, \quad t_0 \leqslant \alpha \leqslant \beta \leqslant 2T - t_0 \\
&\Lambda(t) = \Phi(t,t_0), \quad t \in [t_0, 2T - t_0] \\
&\Phi(t,s) = \Lambda(t)\Lambda^{-1}(s), \quad t_0 \leqslant s \leqslant t \leqslant 2T - t_0
\end{aligned} \tag{6}$$

useful subsequently. For $r \geqslant 0$ we assume

$$\Omega(r) = \max_{x',x'' \in D,\ |x'-x''| \leqslant r} |\varphi(x') - \varphi(x'')| \tag{7}$$

From the continuity of $\varphi(x)$ on D it follows that $\Omega(r) \to 0$ as $r \to +0$. If $\varphi(x)$ satisfies a Lipschitz condition on D, then $\Omega(r) = O(r)$ as $r \to +0$. Let $N \geqslant 1$ be an integer. We set

$$h = \frac{T - t_0}{N}, \quad B_i = \int_{(i-1)h}^{ih} \Phi(T,s) P(s)\, ds, \quad C_i = \int_{(i-1)h}^{ih} \Phi(T,s) Q(s)\, ds \tag{8}$$
$$i = 1, \ldots, N$$

where the integral is understood in the sense usual for the theory of multivalued mappings /10/. We observe that B_i and C_i are nonempty convex compacta. With the number N we associate the quantity

$$\gamma_N = \max_{\eta_1 \in C_1} \min_{\xi_1 \in B_1} \ldots \max_{\eta_N \in C_N} \min_{\xi_N \in B_N} \varphi\Big(\Phi(T,t_0) z_0 + \sum_{i=1}^N (\xi_i + \eta_i)\Big) \tag{9}$$

Using formulas (3), (8), (9), it can be shown that

$$\gamma_N \leqslant \gamma \tag{10}$$

We obtain the estimate $\gamma - \gamma_N$ when $N \geqslant 1$. Let us consider the integral ($v(s) \in Q(s)$, $s \in I$ is an arbitrary measurable function; $v(s-h) = 0 \in R^n$, $t_0 \leqslant s < t_0 + h$)

$$J(v(\cdot)) = \int_{t_0}^T \Lambda^{-1}(s)(v(s) - v(s-h))\, ds \tag{11}$$

We have

$$\begin{aligned}
&\int_{t_0}^T \Lambda^{-1}(s) v(s-h)\, ds = \int_{t_0}^T \Lambda^{-1}(s+h) v(s)\, ds - \int_{T-h}^T \Lambda^{-1}(s+h) v(s)\, ds = \\
&\int_{t_0}^T \Lambda^{-1}(s) v(s)\, ds + \int_{t_0}^T (\Lambda^{-1}(s+h) - \Lambda^{-1}(s)) v(s)\, ds - \int_{T-h}^T \Lambda^{-1}(s+h) v(s)\, ds
\end{aligned} \tag{12}$$

We note that $d(\Lambda^{-1}(t))^*/dt = -A^*(t)(\Lambda^{-1}(t))^*$, where the asterisk denotes transposition. Hence (6) yields the following inequality:

$$|\Lambda^{-1}(s+h) - \Lambda^{-1}(s)| \leqslant \alpha(s,h) = \int_s^{s+h} |A(t)| E(t_0,t)\, dt, \quad t_0 \leqslant s \leqslant T \tag{13}$$

From (2), (6), (11)—(13) it follows that

$$|J(v(\cdot))| \leqslant \beta(h) = \int_{t_0}^T \alpha(s,h) a_2(s)\, ds + \int_{T-h}^T E(t_0, s+h) a_2(s)\, ds \tag{14}$$

We set function $H(s,h)$ equal to the Hausdorff distance between the compacta $Q(s)$ and $Q(s-h)$ for $t_0 \leqslant s \leqslant T$, where $Q(r) = \{0\}$ for $t_0 - h \leqslant r < t_0$. It can be proved that when $s \in I$ the function $H(s,h)$ is Lebesgue-summable and

$$\int_{t_0}^T H(s,h)\, ds \to 0, \quad h \to 0 \tag{15}$$

Obviously

$$Q(s-h) \subset Q(s) + H(s,h) S_1, \quad s \in I \tag{16}$$

where S_1 is the n-dimensional unit ball centered at the origin.

For given $s \in I$, $x \in Q(s-h)$ we consider the following equation relative to $\zeta = (\xi^*; \eta^*)$:

$$\xi + \eta = x, \quad \xi \in Q(s), \quad \eta \in H(s,h) S_1$$

Among the solutions ζ we pick out the lexicographic minimum $\zeta(s, x) = (\xi^*(s, x); \eta^*(s, x))$. For an arbitrary measurable function $v(s) \in Q(s), s \in I$, we set

$$v_0(s, h) = \xi(s, v((s-h)), \quad v(s) = 0 \in R^n, \ t_0 - h \leqslant s < t_0$$

We note that the inequality

$$|v_0(s, h) - v(s-h)| \leqslant H(s, h), \quad s \in I$$

holds on the strength of the definition of $v_0(s, h)$ and of (16). Hence from (6), (11), (14) follows

$$\left|\int_{t_0}^{T} \Phi(T, s)(v(s) - v_0(s, h))\, ds\right| \leqslant \mu(h) = E(t_0, T)\left[\beta(h) + \int_{t_0}^{T} H(s, h) E(t_0, s)\, ds\right] \tag{17}$$

where, by virtue of (13)—(15), each summand within the brackets tends to zero as $h \to 0$. Using formulas (4), (10), (17), we can prove the validity of the inequality

$$\gamma_N \leqslant \gamma \leqslant \gamma_N + \Omega(\mu(h)), \quad h = (T - t_0)/N$$

where $\Omega(r)$, $\mu(h)$ are defined by formulas (7), (17). If the function $A(t)$ is uniformly bounded in norm on I, $Q(t)$ satisfies a Lipschitz condition (in the sense of the Hausdorff metric) on I and the function $\varphi(x)$ satisfies a Lipschitz condition on D (see (5)), then from (7), (13), (14), (17) it follows that $\Omega(\mu(h)) = O(h)$ as $h \to 0$.

REFERENCES

1. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, NAUKA, 1974.
2. CHENTSOV A.G., On a game problem of encounter at a prescribed instant. Mat. Sb., Vol.99, No.3, 1976.
3. CHISTIAKOV S.V., On solving pursuit game problems. PMM Vol.41, No.5, 1977.
4. PSHENICHNYI B.N. and SAGAIDAK M.I., On fixed-time differential games. KIBERNETIKA, No.2, 1970.
5. CHERNOUS'KO F.L. and MELIKIAN A.A., Game Problems of Control and Search. Moscow, NAUKA, 1978.
6. PETROSIAN L.A., Differential Pursuit Games. Leningrad. Izd. Leningradsk. Gos. Univ., 1977.
7. FRIEDMAN A., Differential Games. New York: Wiley-Interscience, 1971.
8. POLISHCHUK E.G. Upper estimate for the value of a linear differential game. PMM Vol.44, No.2, 1980.
9. NIKOL'SKII M.S., On certain fixed-time differential games. Dokl. Akad. Nauk SSSR, Vol.240, No.2, 1978.
10. CASTAING Ch. Sur les multi-applications measurables. Rev. franc. inform. et rech. opér., No.1, 1967.
11. DEMIDOVICH B.P., Lectures on the Mathematical Theory of Stability. Moscow, NAUKA, 1967.

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